DETERMINATION OF THE GROUP OF RATIONALITY OF A LINEAR DIFFERENTIAL EQUATION.

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1. When a special algebraic equation is given, its group G can be theoretically determined: 1° by constructing an $n!$-valued function and finding the Galois resolvant; or, 2°, by applying the characteristic double-property of the group, that every rational function of the roots which remains uualtered by all the substitutions of G lies in the domain of rationality; and conversely. Both of these methods are generally impracticable, and in practice some device is resorted to in order to obtain the group. When a single rational function is known the following theorem has been found very useful:* "If a rational function $\psi(x_1 \ldots . x_n)$ remains formally unaltered by the substitution of a group G' and by no other substitutions, and if ψ equals a quantity lying in the domain of rationality R, and if the conjugates of ψ under $G_{n!}$ are all distinct, then the group of the given equation for the domain R is a subgroup of G'."

A similar situation confronts us when we deal with linear differential equations. When a special linear differential equation is given, its group may be found: 1° by constructing the Picard resolvant†, or, 2° by applying the Picard-Vessiot characteristic double-property of the group‡, that every rational differential function (of a fundamental system) of the integral which remains unaltered, as a function of x, by all the transformations of G lies in R (the domain of rationality); and conversely. As in the case of algebraic equations both of these methods are impracticable, and in order to find the group a device must be resorted to. With this end in view, I will prove the analog of the above theorem of Algebra, viz: *If a rational differential function (of a fundamental system) of integrals $\psi(y_1 \ldots . y_n, y'_1 \ldots . y'_n, \ldots .)$ remains formally unaltered by the transformations of a complex r-parameter linear homogeneous group G_r; and if, when ψ is transformed by the most general linear homogeneous transformation, it depends on n^2-r essential parameters§, then the group G of the given equation is a subgroup of G_r.*

The advantage of this theorem is that only one function ψ is required, whereas the Picard-Vessiot double theorem relates to the entire infinity of rational functions.

2. Let the given differential equation be

$$(1) \qquad \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots . + p_n y = 0$$

* L. E. Dickson, *Theory of Algebraic Equations* (John Wiley and Sons).

† Schlesinger's *Handbuch der Theorie der linearen Differentialgleichungen*, II, part 1, pp. 60 and 68.

‡ Schlesinger, *Handbuch*, II, 1, p. 71.

§That is to say, if $G_{n^2} : \bar{y}_i = \sum_{k=1}^{n} a_{ik} y_k$ is the most general transformation, then $\psi(\bar{y}) = \psi(\Sigma a_{ik} y_k) = \phi(y, A_1 \ldots . A_{n^2-r})$ where the n^2-r A's are all essential. Cf. Picard, *Traité d' Analyse*, III, p. 508; also p. 519.

which is supposed to be special, *i. e.* the coefficients p are known (and not indeterminate) functions of x. A domain of rationality R is supposed to be assigned, this domain containing at least the coefficients. The group of rationality of the differential equation (1) in (R) is in general a complex group made up of ν systems of linear homogeneous transformations

$$G_r : \overline{y}_{ij} = \sum_{k=1}^{n} a_{ikj} y_{kj} \qquad \begin{pmatrix} i=1\ldots.n \\ j=0,\ 1\ldots.\nu-1 \end{pmatrix}.$$

$\psi(y)=a(x)$ is *formally* invariant under this r-parameter group. This group G_r contains a certain maximal continuous subgroup Γ which is generated by infinitesimal transformations

$$\Gamma : \overline{y}_{io} = \sum_{k} a_{iko} y_{ko} \qquad (i=1\ldots.n).$$

If $T_0=I,\ T_1,\ \ldots.T_{\nu-1}$ are the transformations for which

$$T_j^{-1}\, \Gamma\, T_j = \Gamma \qquad (j=0,\ 1\ldots.\nu-1),$$

then G_r is of the form* $G_r : \Gamma,\ T_1\Gamma,\ \ldots.,\ T_{\nu-1}\Gamma$.

Moreover, $T_0=I,\ T_1\ldots.,\ T_{\nu-1}$ form a substitution-group.

Suppose now that $\phi(y)=r(x)$ is any differential function (as usual, of a fundamental system) of integrals of equation (1), which remains *numerically* unaltered (*i. e.* as function of x) under G_r. It follows that $\phi(y)$ is *formally* unaltered under Γ.† Under the group G_r, ϕ will therefore take ν values: $\phi=\phi_0,\ \phi_1 \ldots.\phi_{\nu-1}$, and the function

$$\phi = \frac{1}{\nu}(\phi_0 + \phi_1 + \ldots. + \phi_{\nu-1})$$

remains formally invariant under G_r. It follows now by the Lagrange-Vessiot theorem‡ that ϕ is a rational function of ψ and is therefore a quantity lying in R (*i. e.* $r(x)$ lies in R).

3. We know now that our group G_r has the property that every rational differential function of a fundamental system of integrals $y_1,\ \ldots.,\ y_n$, which remains numerically unaltered by the most general transformation of G_r lies in the domain of rationality R. We will next show that G_r contains as a subgroup (or coincides with) the group of rationality G of the equation (1). Writing

$$(2) \qquad u(y) = A_1 y_1 + \ldots. + A_n y_n$$

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* Schlesinger, *Handbuch*, II, 1, p. 79.

† Schlesinger, *Handbuch*, II, 1, p. 79, calls θ a characteristic invariant of Γ; cf. also Gino Fano, *Mathematische Annalen*, Vol. 53, p. 403.

‡ Vessiot: *Annales de l' Ecole Normale Superieure*, 1892, p. 223; Schlesinger, II, 1, pp. 53, 57; Picard, *Traite d' Analyse*, III, pp. 521-2.

(the A's being undetermined functions of x) we differentiate n^2 times and eliminate the derivatives of the nth and higher orders by means of the differential equation (1). This gives us n^2+1 equations between the n^2 quantities $y_1 \ldots . y_n$, $y'_1 \ldots . y'_n$, $y_1^{(n-1)} \ldots . y_n^{(n-1)}$; eliminating we obtain a linear differential equation

$$(3) \qquad \frac{d^{n^2}u}{dx^{n^2}} + \pi_1 \frac{d^{n^2-1}u}{dx^{n^2-1}} + \ldots . + \pi_{n^2} u = 0.$$

We assume that

$$(4) \qquad \theta(u) = 0$$

is a non-linear differential equation of the lowest order with the following properties: its coefficients are rational functions of the A's, their derivatives, and of x; it is irreducible in the sense of Koenigsberger;* at least one of its integrals satisfies (3).†

The equation (2) was differentiated n^2 times; neglecting the results of the last differentiation, we have n^2 equations in the n^2 quantities $y_1 \ldots . y_n$, $y'_2 \ldots . y'_n$, $\ldots .;$ $y_1^{(n-1)} \ldots . y_n^{(n-1)}$. Solving these equations, we obtain

$$y_j^{(i)} = w_{ji1} u + w_{ji2} \frac{du}{dx} + \ldots . + w_{jin^2} \frac{d^{n^2-1}u}{dx^{n^2-1}}.$$

Substituting the values thus obtained in $\phi(y_1 \ldots . y_n, \ldots ., y_1^{(n-1)} \ldots . y_n^{(n-1)}) = r(x)$ we obtain a rational differential expression

$$(5) \qquad F(u, \frac{du}{dx} \ldots . \frac{d^\nu u}{dx^\nu}) = r(x) \qquad (\nu \overline{\gtrless} n^2-1).$$

The equation (4) having an integral in common with (5), all of the other integrals of (4) must also satisfy (5).‡ Thus, when we replace u in (5) by any integral of (4), F remains always equal to $r(x)$. The group of rationality of (1) is the totality of transformations which correspond to the passage from some particular integral u_1 of

$$(4) \qquad \theta(u) = 0$$

to all the other integrals of this equation.§ We saw however that $F = r(x)$ for all the integrals of $\theta(u)=0$, hence $\phi(y)$ is numerically invariant under the group of rationality of the equation (1). If

$$\theta(u) \equiv F(u) - r(x) = 0$$

* Schlesinger, *Handbuch*, II, 1, p. 66.

† *Handbuch*, II, 1, p. 65. In a letter to Picard (*Bulletin des Sciences Mathem.*, March, 1902) Professor Alfred Loewy proved the important theorem that the various equations (4) which are satisfied by the fundamental integrals of (3) are all of the *same* order.

‡ Since (4) is irreducible. Cf. Picard, *Traite*, III, p. 525.

§ *Handbuch*, II, 1, pp. 68-9.

G and G_r will coincide, but in general F may well be of higher order or higher degree than θ and consequently G will be a subgroup of G_r.

4. In the enunciation of the theorem I added that when $\psi(y)$ was transformed by the most general linear homogeneous transformation (in n variables and n^2 parameters) it would become a function of n^2-r essential parameters. This corresponds to the analogous requirement in Algebra that the conjugates of $\psi(x_1 \ldots . x_n)$ under the total symmetric group shall be distinct. The reason for this condition is that it enters in the proof of the Lagrange-Vessiot theorem which was employed in 2.

5. Example 1°: Consider the equation

$$\frac{d^2y}{dx^2}+y=0. \tag{6}$$

A fundamental pair of integrals is

$$y_1=\sin x,\quad y_2=\cos x;$$

between these exists the relation

$$\psi(y)\equiv y_1^2+y_2^2=1. \tag{7}$$

Now $\psi(y)$ remains formally unaltered under the group G_1:

$$\begin{cases}\bar{y}_1=y_1\cos a-y_2\sin a,\\ \bar{y}_2=y_1\sin a+y_2\cos a;\end{cases}\qquad \begin{cases}\bar{y}_1=y_1\cos a+y_2\sin a,\\ \bar{y}_2=-y_1\sin a+y_2\cos a.\end{cases}$$

When ψ is transformed by

$$\bar{y}_1=a_{11}y_1+a_{12}y_2,\quad \bar{y}_2=a_{21}y_1+a_{22}y_2,$$

there results $\psi(\bar{y})\equiv(a_{11}y_1+a_{12}y_2)^2+(a_{21}y_1+a_{22}y_2)^2$, a function of *three* (n^2-r) essential parameters, namely, $A_1y_1^2+A_2y_2^2+A_3y_1y_2$, where $A_1=a_{11}^2+a_{21}^2$, $A_2=a_{12}^2+a_{22}^2$, $A_3=2(a_{11}a_{12}+a_{21}a_{22})$. The group of rationality of (6) is therefore either G_1 or the identity. In the domain* of the trigonometric functions (or their equivalents, such as the exponential functions) the group of rationality of (1) is the identity. In a domain not including the trigonometric functions, G_1 is the required group.

Example 2°: The question of numerical invariance of the rational functions of the integrals was first raised by Klein.† Previous to this Picard‡ had given:

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* For differential equations the domain of rationality is composed of the four fundamental operations, extraction of roots, differentiation and any functions which may be decided on arbitrarily. The functions are however so chosen as to always make the coefficients of the given equation rational.

† F. Klein, *Hohere Geometrie*, II, pp. 298-9. (B. G. Teubner, Leipzig).

‡ Picard, *Comptes Rendus*, 1883, p. 1134. Picard writes the second equation $Y_2=\sqrt{(1-\lambda^2)}y_1-\lambda y_2$; this cannot be correct since $\lambda=1$ gives $Y_2=-y_2$ instead of the identical transformation $Y_2=y_2$.

$$Y_1=\lambda y_1+\sqrt{(1-\lambda^2)}y_2$$
$$Y_2=-\sqrt{(1-\lambda^2)}y_1+\lambda y_2$$

as the group of the equation

$$(8) \qquad x(1-x)\frac{d^2y}{dx^2}-\frac{x}{2}\frac{dy}{dx}+a^2y=0, \qquad (a=\text{constant})$$

since the relation between a suitably selected pair of fundamental integrals is $y_1^2+y_2^2=1$.*

On the basis of our theorem however, G_r becomes G_1 :

$$\begin{cases} Y_1=\lambda y_1+\sqrt{(1-\lambda^2)}y_2, \\ Y_2=-\sqrt{(1-\lambda^2)}y_1+\lambda y_2, \end{cases} \qquad \begin{cases} \overline{Y}_1=\lambda y_1-\sqrt{(1-\lambda^2)}y_2, \\ \overline{Y}_2=\sqrt{(1-\lambda^2)}y_1+\lambda y_2 ; \end{cases}$$

and the group of rationality of (8) will be G_1 or the identity according to the domain of rationality which is selected. Since (8) is a special case of the hypogeometric equation

$$x(1-x)\frac{d^2y}{dx^2}+[\Upsilon-x(1+a+\beta)]\frac{dy}{dx}-a\beta y=0$$

the adjunction of the hypogeometric function would reduce the group of rationality to the identical transformation.

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* Picard, *loc. cit.*

SOME FALLACIES IN TEXT-BOOKS ON ELEMENTARY SOLID GEOMETRY.

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So much has been written on the fallacies encountered in elementary plane geometries in applying to curves theorems on the lengths of broken lines, that it is, perhaps, superfluous to note the more frequent errors of similar character in elementary texts on solid geometry.

For example, in the case of a cylindrical surface, there is in its definition no connection between it and a prismatic surface; and any attempt to *prove* that the surface of the inscribed or circumscribed prism approaches the surface of the cylinder as the number of sides is indefinitely increased, seems fallacious. For no matter how great is the number of sides of the inscribed prism, there is still an *infinite gap* between it and the cylinder in which it is inscribed, and to assume